



Munich Personal RePEc Archive

# Individual Risk and Lebesgue Extension without Aggregate Uncertainty

Yeneng Sun and Yongchao Zhang

29. February 2008

Online at <http://mpra.ub.uni-muenchen.de/7448/>

MPRA Paper No. 7448, posted 5. March 2008 14:03 UTC

# Individual Risk and Lebesgue Extension without Aggregate Uncertainty\*

Yeneng Sun<sup>†</sup> and Yongchao Zhang<sup>‡</sup>

First version: December 2007; this version: February 2008

## Abstract

Many economic models include random shocks imposed on a large number (continuum) of economic agents with individual risk. In this context, an exact law of large numbers and its converse is presented in [23] to characterize the cancelation of individual risk via aggregation. However, it is well known that the Lebesgue unit interval is not suitable for modeling a continuum of agents in the particular setting. The purpose of this note is to show that an extension of the Lebesgue unit interval does work well as an agent space with various desirable properties associated with individual risk.

**Keywords:** No aggregate uncertainty, independence, exact law of large numbers, Fubini extension, Lebesgue measure.

---

\*The authors are grateful to Bob Anderson, Jerry Keisler, Ali Khan, Rich McLean and Rabee Tourky for the encouragement to us for working on the problem studied here. They also thank Peter Loeb for some useful discussions on the work of Fremlin.

<sup>†</sup>Department of Economics, National University of Singapore, 1 Arts Link, Singapore, 117570.  
Email: ynsun@nus.edu.sg

<sup>‡</sup>Department of Mathematics, National University of Singapore, Science Drive 2, Singapore, 117543.  
Email: yongchao@nus.edu.sg

# 1 Introduction

Models with a continuum of agents are widely used in economics. One often chooses to work with the the unit interval with the Lebesgue measure as the agent space. However, it was already noted by Aumann that the choice of the Lebesgue unit interval as a model for the agent space is of no particular significance and any atomless probability space is precisely what is needed to ensure that each individual agent has no influence.<sup>1</sup>

Many economic models have also been based on a continuum of agents with individual risk. Formally, a continuum of independent random variables is used to model individual level random shocks imposed on a large number of economic agents. The desirable result is an exact law of large numbers which guarantees the cancelation of individual risk at the aggregate level.<sup>2</sup> It is shown in [23] that a process measurable in a Fubini extension is essentially pairwise independent if and only if it satisfies the property of coalitional aggregate certainty.<sup>3</sup> The latter means that aggregation at the coalitional level removes uncertainty.

Section 5 of [23] considers the existence of a Fubini extension that allows one to construct processes with essentially pairwise independent random variables taking any given variety of distributions. Many probability spaces can be used as the relevant index space; for example, the big class of atomless Loeb probability spaces (see [18]). One can also work with an index space based on some atomless measure space on the unit interval  $[0, 1]$ .

However, unlike the case of a continuum of agents in a deterministic model, it is well known that an economic model with an i.i.d. process based the classical continuum product space and indexed by the Lebesgue unit interval has the sample measurability problem.<sup>4</sup> Moreover, Corollary 4.3 of [23] shows that under the framework of a Fubini extension, almost all sample functions of any process with essentially pairwise independent random variables cannot be Lebesgue measurable. It is also pointed out by Feldman and Gilles in Section 1 of [9] that a continuum of i.i.d. Bernoulli random variables indexed by the Lebesgue unit interval cannot satisfy the the property of coalitional aggregate certainty.

Since the Lebesgue unit interval is the simplest atomless probability space and it is not suitable for modeling a continuum of agents with individual risk, a natural question is whether one can find some extension of the Lebesgue unit interval as the agent space with the desired property. The purpose of this note is to provide a positive answer to that question. In particular, we construct essentially pairwise independent processes measurable in a Fubini extension, where the sample functions are measurable with respect to some extension of the Lebesgue unit interval and the random variables can take any given variety of distributions. It follows immediately from the exact law of large numbers

---

<sup>1</sup>For this point, see p. 44 of [3]. For the discussion of various other formulations of negligible agents, see [16].

<sup>2</sup>See [23] for many earlier references on this. For some more recent applications of the law of large numbers, see [6], [7], [17], [19], [20], [24], [25].

<sup>3</sup>See Definitions 1 and 2 below respectively for the precise meaning of essential pairwise independence and coalitional aggregate certainty. The equivalence result is shown in Theorem 2.8 of [23].

<sup>4</sup>See [4], [5], [14] and the detailed discussion in [23].

that the type of result as mentioned in Section 1 of Feldman and Gilles [9] holds for some extension of the Lebesgue unit interval. The point is that though the Lebesgue unit interval fails to be an agent space modeling a continuum of agents with individual risk, some extension of it does work.

The rest of the paper is organized as follows. Section 2 presents some basic definitions and a previous characterization result on the cancelation of individual risk via aggregation. The main result is stated in section 3. The proofs are given in the Appendix.

## 2 Basics

Let  $(I, \mathcal{I}, \lambda)$  be an atomless probability space which is used to model the agent space of many economic agents. In our setting, it will be the parameter space for a process. Let  $(\Omega, \mathcal{F}, P)$  be a sample probability space, which models the space of uncertain states of the world. A process  $f$  from  $I \times \Omega$  to a complete separable metric space  $X$  with Borel  $\sigma$ -algebra  $\mathcal{B}$  is a mapping from  $I \times \Omega$  to  $X$  such that (1) for  $\lambda$ -almost all  $i \in I$ , the random shock  $f_i$  imposed on agent  $i$  is a random variable defined on  $(\Omega, \mathcal{F}, P)$  whose distribution  $Pf_i^{-1}$  on  $X$  is defined by  $Pf_i^{-1}(B) = P[f_i^{-1}(B)]$  for each  $B \in \mathcal{B}$ ; (2) for every  $B \in \mathcal{B}$ , the mapping  $i \mapsto Pf_i^{-1}(B)$  is  $\mathcal{I}$ -measurable.

The meaning of individual risk is that each individual agent is allowed to have correlation with a negligible group of other agents. This is formalized as the concept of essential pairwise independence.

**Definition 1** *A process  $f$  from  $I \times \Omega$  to a complete separable metric space  $X$  is said to be essentially pairwise independent if for  $\lambda$ -almost all  $s \in I$ , the random variables  $f_s$  and  $f_i$  are independent for  $\lambda$ -almost all  $i \in I$ .*

A desirable result for individual risk is its cancelation at the aggregate level. That is, aggregation over a non-negligible group of agents leads to no uncertainty. The following is a formal definition of coalitional aggregate certainty.

**Definition 2** *Let  $f$  be a process from  $I \times \Omega$  to a complete separable metric space  $X$ . For any coalition  $S$  (i.e.,  $S \in \mathcal{I}$  with  $\lambda(S) > 0$ ), let  $f^S$  be the restriction of  $f$  to  $S \times \Omega$ ,  $\mathcal{I}^S = \{C \in \mathcal{I} : C \subseteq S\}$ , and  $\lambda^S$  the probability measure rescaled from the restrictions of  $\lambda$  to  $\mathcal{I}^S$ . The process  $f$  is said to satisfy the property of coalitional aggregate certainty if for  $P$ -almost all  $\omega \in \Omega$ , the sample function  $f_\omega$  is  $\mathcal{I}$ -measurable, and for each coalition  $S$ , the empirical (sample) distribution  $\lambda(f_\omega^S)^{-1}(\cdot)$  is  $\int_S Pf_i^{-1}(\cdot) d\lambda^S$  for  $P$ -almost all  $\omega \in \Omega$ .*

When  $X$  is the real line  $\mathbb{R}$  and the random variables  $f_i$  are i.i.d. with a common distribution function  $F$ , coalitional aggregate certainty means that for each coalition  $S$ , the empirical distribution function  $F_\omega^S$  generated by the restricted sample function  $f_\omega^S$  is  $F$  for  $P$ -almost all  $\omega \in \Omega$ . It is easy to construct examples of a continuum of independent random variables with this aggregation property for the grand coalition  $I$  or for all the coalitions; see the discussion in Section 6.3 of [23], and [2], [12], [14]. However, one can also construct other examples of a continuum of independent random variables whose sample functions may not be measurable, or behave in a very “strange” way. In fact, for

an i.i.d. process based on the usual continuum product via the Kolmogorov construction, one can obtain the absurd claim that almost all sample functions are essentially equal to an arbitrarily given function  $h$  on the index space (see Proposition 6.1 of [23]); and thus, the sample distribution can be undefined or completely arbitrary (see also [4] and [14]).

The main difficulty for working with an essentially pairwise independent process  $f$  is that if it is jointly measurable with respect to the usual product  $\sigma$ -algebra  $\mathcal{I} \otimes \mathcal{F}$ , then the random variables  $f_i$  are essentially constant for almost all  $i \in I$  (see Proposition 2.1 of [23]). Consequently, the usual product probability space  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$  will (typically) be inadequate to prove any meaningful result on no aggregate uncertainty. As shown in [23], a simple way to resolve this problem is to work with an extension of the usual product probability space that retains the Fubini property.

**Definition 3** *Let  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$  be the usual product probability space of the two probability spaces  $(I, \mathcal{I}, \lambda)$  and  $(\Omega, \mathcal{F}, P)$ . A probability space  $(I \times \Omega, \mathcal{W}, Q)$  extending  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$  is said to be a Fubini extension if for any real-valued  $Q$ -integrable function  $f$  on  $(I \times \Omega, \mathcal{W})$ ,*

*(1) the two functions  $f_i$  and  $f_\omega$  are integrable respectively on  $(\Omega, \mathcal{F}, P)$  for  $\lambda$ -almost all  $i \in I$ , and on  $(I, \mathcal{I}, \lambda)$  for  $P$ -almost all  $\omega \in \Omega$ ;*

*(2)  $\int_\Omega f_i dP$  and  $\int_I f_\omega dP$  are integrable respectively on  $(I, \mathcal{I}, \lambda)$  and  $(\Omega, \mathcal{F}, P)$ , with  $\int_{I \times \Omega} f dQ = \int_I (\int_\Omega f_i dP) d\lambda = \int_\Omega (\int_I f_\omega d\lambda) dP$ .<sup>5</sup>*

*To reflect the fact that the probability space  $(I \times \Omega, \mathcal{W}, Q)$  has  $(I, \mathcal{I}, \lambda)$  and  $(\Omega, \mathcal{F}, P)$  as its marginal spaces, as required by the Fubini property, it will be denoted by  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ .*

The following result is shown in Theorem 2.8 of [23]. It indicates that the framework of Fubini extension does deliver the desired exact law of large numbers which guarantees the cancelation of individual risk at the aggregate level.

**Lemma 1** *Let  $f$  be a measurable process from a Fubini extension  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  to a complete separable metric space  $X$ . Then  $f$  satisfies the property of coalitional aggregate certainty if and only if  $f$  is essentially pairwise independent.*

### 3 The main result

Let  $L = [0, 1]$ ,  $\mathcal{L}$  the  $\sigma$ -algebra of Lebesgue measurable sets, and  $\eta$  the Lebesgue measure defined on  $\mathcal{L}$ . The Lebesgue unit interval is simply  $(L, \mathcal{L}, \eta)$ .

We are now ready to state the main result of this paper, which shows that some extension  $(I, \mathcal{I}, \lambda)$  of the Lebesgue unit interval  $(L, \mathcal{L}, \eta)$  can be used as the agent space

---

<sup>5</sup>The classical Fubini Theorem is only stated for the usual product measure spaces. It does not apply to integrable functions on  $(I \times \Omega, \mathcal{W}, Q)$  since these functions may not be  $\mathcal{I} \otimes \mathcal{F}$ -measurable. However, the conclusions of that theorem do hold for processes on the enriched product space  $(I \times \Omega, \mathcal{W}, Q)$  that extends the usual product.

modeling a continuum of agents with individual risk in a very general setting.<sup>6</sup> In particular, we show the existence of essentially pairwise independent processes measurable in a Fubini extension, where the sample functions are measurable with respect to the extended Lebesgue interval  $(I, \mathcal{I}, \lambda)$  and the random variables can take any given variety of distributions.

**Theorem 1** *Let  $I$  be the unit interval  $[0, 1]$  and  $X$  be a complete separable metric space. There exists a probability space  $(I, \mathcal{I}, \lambda)$  extending the Lebesgue unit interval, a probability space  $(\Omega, \mathcal{F}, P)$ , and a Fubini extension  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  such that for any measurable mapping  $\varphi$  from  $(I, \mathcal{I}, \lambda)$  to the space  $\mathcal{M}(X)$  of Borel probability measures<sup>7</sup> on  $X$ , there is a  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process  $f$  from  $I \times \Omega$  to  $X$  such that the random variables  $f_i$  are essentially pairwise independent, and the distribution  $P f_i^{-1}$  is the given distribution  $\varphi(i)$  for  $\lambda$ -almost all  $i \in I$ .*

The following corollary on the special case of an i.i.d. process is obvious.

**Corollary 1** *Let  $I$  and  $X$  be as in Theorem 1. There exists a probability space  $(I, \mathcal{I}, \lambda)$  extending the Lebesgue unit interval, a probability space  $(\Omega, \mathcal{F}, P)$ , and a Fubini extension  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  such that for any Borel probability measure  $\tau$  on  $X$ , there is a  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process  $f$  from  $I \times \Omega$  to  $X$  such that the random variables  $f_i$  are essentially pairwise independent with common distribution  $\tau$ .*

**Remark 1** *By Lemma 1, essential pairwise independence implies coalitional aggregate certainty. Thus the processes in Theorem 1 and Corollary 1 satisfy coalitional aggregate certainty. For the special case that  $X$  is the real line  $\mathbb{R}$  and the random variables  $f_i$  are i.i.d. with a common distribution function  $F$  as in Corollary 1, we obtain that for each coalition  $S$ , the empirical distribution function  $F_\omega^S$  generated by the restricted sample function  $f_\omega^S$  is  $F$  for  $P$ -almost all  $\omega \in \Omega$ . Thus, the type of result as mentioned in Section 1 of Feldman and Gilles [9] holds for some extension of the Lebesgue unit interval.*

## 4 Appendix

In this appendix, the unit interval  $[0, 1]$  will have a different notation in a different context. Recall that  $(L, \mathcal{L}, \eta)$  is the Lebesgue unit interval. We shall often work with the case that the target space  $X$  is the unit interval  $[0, 1]$  with uniform distribution  $\mu$ . Here  $\mu$  is simply the Lebesgue measure defined on Borel  $\sigma$ -algebra  $\mathcal{B}$  of  $[0, 1]$ . Note that the Lebesgue measure defined on the Lebesgue  $\sigma$ -algebra  $\mathcal{L}$  is denoted by  $\eta$ .

The following result is Proposition 5.6 of [23].

**Lemma 2** *There is an atomless probability space  $(K, \mathcal{K}, \kappa)$  with  $K = [0, 1]$ , a probability space  $(\Omega, \mathcal{F}, P)$ , a Fubini extension  $(K \times \Omega, \mathcal{K} \boxtimes \mathcal{F}, \kappa \boxtimes P)$ , and a  $\mathcal{K} \boxtimes \mathcal{F}$ -measurable process  $g$  from  $K \times \Omega$  to  $[0, 1]$  such that the random variables  $g_k$  are pairwise independent and identically distributed (i.i.d.) with common uniform distribution  $\mu$  on  $[0, 1]$ .*

<sup>6</sup> $(I, \mathcal{I}, \lambda)$  extending  $(L, \mathcal{L}, \eta)$  means that  $I = L = [0, 1]$ ,  $\mathcal{I}$  contains  $\sigma$ -algebra  $\mathcal{L}$  of Lebesgue measurable sets, and  $\lambda$  extends the Lebesgue measure  $\eta$  on  $\mathcal{L}$ .

<sup>7</sup> $\mathcal{M}(X)$  is endowed with the topology of weak convergence of measures. The measurability of  $\varphi$  is equivalent to the measurability of the mappings  $i \mapsto \varphi(i)(B)$  for  $B \in \mathcal{B}$ .

In addition, in Proposition 5.6 of [23], the sample probability space  $(\Omega, \mathcal{F}, P)$  is an extension of the usual continuum product; the index space  $(K, \mathcal{K}, \kappa)$  is obtained from a Loeb probability space via a bijection. As shown in Corollary 3 of [15],  $(K, \mathcal{K}, \kappa)$  is not an extension of the Lebesgue unit interval  $(L, \mathcal{L}, \eta)$ . However, as mentioned earlier, the purpose of this paper is to obtain some extension of the Lebesgue unit interval as an agent space with various desirable properties associated with individual risk.

The following is essentially taken from Lemma 419I of Fremlin [10]. Its proof is based on the Transfinite Induction.

**Lemma 3** *There is a disjoint family  $\mathcal{C} = \{C_k : k \in K = [0, 1]\}$  of subsets of  $L = [0, 1]$  such that  $\bigcup_{k \in K} C_k = L$ , and for each  $k \in K$ ,  $\eta_*(C_k) = 0$  and  $\eta^*(C_k) = 1$ , where  $\eta_*$  and  $\eta^*$  are the respective inner and outer measures of the Lebesgue measure  $\eta$ .*

The original version of Lemma 419I of Fremlin [10] does not require  $\bigcup_{k \in K} C_k = L$ . Suppose  $\bigcup_{k \in K} C_k \neq L$ ; let  $B = L \setminus \bigcup_{k \in K} C_k$ . Since the cardinality of  $B$  is at most the cardinality of  $[0, 1]$ , we can redistribute at most one point of  $B$  into each  $C_k$  in the family  $\mathcal{C}$ .

As in the proof of Lemma 521P(b) of [11], we define a subset  $C$  of  $L \times K$  by letting  $C = \{(l, k) \in L \times K : l \in C_k, k \in K\}$ . Let  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$  be the usual product probability space. For any  $\mathcal{L} \otimes \mathcal{K}$ -measurable set  $U$  that contains  $C$ ,  $C_k \subseteq U_k$  for each  $k \in K$ . The Fubini property of  $\eta \otimes \kappa$  implies that for  $\kappa$ -almost all  $k \in K$ ,  $U_k$  is  $\mathcal{L}$ -measurable, which means that  $\eta(U_k) = 1$  (since  $\eta^*(C_k) = 1$ ). Since  $\eta \otimes \kappa(U) = \int_K \eta(U_k) d\kappa$ , we have  $\eta \otimes \kappa(U) = 1$ . Therefore, the  $\eta \otimes \kappa$ -outer measure of  $C$  is one.

Since the  $\eta \otimes \kappa$ -outer measure of  $C$  is one, the method in [5] (see p. 69) can be used to extend  $\eta \otimes \kappa$  to a measure  $\gamma$  on the  $\sigma$ -algebra  $\mathcal{U}$  generated by the set  $C$  and the sets in  $\mathcal{L} \otimes \mathcal{K}$  with  $\gamma(C) = 1$ . It is easy to see that  $\mathcal{U} = \{(U_1 \cap C) \cup (U_2 \setminus C) : U_1, U_2 \in \mathcal{L} \otimes \mathcal{K}\}$ , and  $\gamma[(U_1 \cap C) \cup (U_2 \setminus C)] = \eta \otimes \kappa(U_1)$  for any measurable sets  $U_1, U_2 \in \mathcal{L} \otimes \mathcal{K}$ . Let  $\mathcal{T}$  be the  $\sigma$ -algebra  $\{U \cap C : U \in \mathcal{L} \otimes \mathcal{K}\}$ , which is the collection of all the measurable subsets of  $C$  in  $\mathcal{U}$ . The restriction of  $\gamma$  to  $(C, \mathcal{T})$  is still denoted by  $\gamma$ . Then,  $\gamma(U \cap C) = \eta \otimes \kappa(U)$ , for every measurable set  $U \in \mathcal{L} \otimes \mathcal{K}$ . Note that  $(L \times K, \mathcal{U}, \gamma)$  is an extension of  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$ .

Consider the projection mapping  $\pi : L \times K \rightarrow L$  with  $\pi(l, k) = l$ . Let  $\psi$  be the restriction of  $\pi$  to  $C$ . Since the family  $\mathcal{C}$  is a partition of  $L = [0, 1]$ ,  $\psi$  is a bijection between  $C$  and  $L$ . It is obvious that  $\pi$  is a measure-preserving mapping from  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$  to  $(L, \mathcal{L}, \eta)$ , i.e., for any  $B \in \mathcal{L}$ ,  $\pi^{-1}(B) \in \mathcal{L} \otimes \mathcal{K}$  and  $\eta \otimes \kappa[\pi^{-1}(B)] = \eta(B)$ ; and thus  $\pi$  is a measure-preserving mapping from  $(L \times K, \mathcal{U}, \gamma)$  to  $(L, \mathcal{L}, \eta)$ . Since  $\gamma(C) = 1$ ,  $\psi$  is a measure-preserving mapping from  $(C, \mathcal{T}, \gamma)$  to  $(L, \mathcal{L}, \eta)$ , i.e.,  $\gamma[\psi^{-1}(B)] = \eta(B)$  for any  $B \in \mathcal{L}$ .

To introduce one more measure structure on  $[0, 1]$ , we shall also denote it by  $I$ . Let  $\mathcal{I}$  be the  $\sigma$ -algebra  $\{S \subseteq I : \psi^{-1}(S) \in \mathcal{T}\}$ . Define a set function  $\lambda$  on  $\mathcal{I}$  by letting  $\lambda(S) = \gamma[\psi^{-1}(S)]$  for each  $S \in \mathcal{I}$ . Since  $\psi$  is a bijection,  $\lambda$  is a well-defined probability measure on  $(I, \mathcal{I})$ . Moreover,  $\psi$  is also an isomorphism from  $(C, \mathcal{T}, \gamma)$  to  $(I, \mathcal{I}, \lambda)$ . Since  $\psi$  is a measure-preserving mapping from  $(C, \mathcal{T}, \gamma)$  to  $(L, \mathcal{L}, \eta)$ , it is obvious that  $(I, \mathcal{I}, \lambda)$  is an extension of the Lebesgue unit interval  $(L, \mathcal{L}, \eta)$ .<sup>8</sup>

<sup>8</sup>A similar observation was made in §6 of [21] for the case of the product of the Lebesgue unit interval

As we have seen, based on the constructions as used in Lemma 419I of Fremlin [10] and Lemma 521P(b) of [11], it is rather straightforward to construct  $(I, \mathcal{I}, \lambda)$ . To prove Theorem 1, the key part is to construct a Fubini extension and essentially pairwise independent measurable processes whose random variables take any variety of distributions. We shall first consider a special case of Theorem 1 below.

**Proposition 1** *There is a Fubini extension  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  and an essentially pairwise independent process  $f : I \times \Omega \rightarrow [0, 1]$  such that  $f$  is  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable, and for each  $i \in I$ , the distribution of the random variable  $f_i$  is the uniform distribution  $\mu$  on  $[0, 1]$ .*

**Proof:** We construct the process  $f : I \times \Omega \rightarrow [0, 1]$  in three steps.

**Step 1.** Based on the process  $g$  and the Fubini extension  $(K \times \Omega, \mathcal{K} \boxtimes \mathcal{F}, \kappa \boxtimes P)$  in Lemma 2, we construct a new process  $G$  from the triple product space  $L \times K \times \Omega$  to  $[0, 1]$  with  $G(l, k, \omega) = g(k, \omega)$  for each  $(l, k, \omega) \in L \times K \times \Omega$ . Here the index space is augmented to the usual product space  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$  while the sample space remains  $(\Omega, \mathcal{F}, P)$ .

For each  $(l, k) \in L \times K$ ,  $G_{(l, k)} = g_k$  is a random variable on the sample space with common uniform distribution  $\mu$  on  $[0, 1]$ . Moreover, the process  $G$  is essentially pairwise independent. In fact, for any  $(l_0, k_0) \in L \times K$ , if  $k \neq k_0$ ,  $g_{k_0}$  and  $g_k$  are independent random variables, so are the random variables  $G_{(l_0, k_0)} = g_{k_0}$  and  $G_{(l, k)} = g_k$ . It is obvious that the subset  $\{(l, k) \in L \times K : k \neq k_0\}$  has full  $\eta \otimes \kappa$ -measure.

Now consider the usual product space  $(L \times K \times \Omega, \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F}), \eta \otimes (\kappa \boxtimes P))$  of the Lebesgue unit interval  $(L, \mathcal{L}, \eta)$  with the Fubini extension  $(K \times \Omega, \mathcal{K} \boxtimes \mathcal{F}, \kappa \boxtimes P)$ . Note that the process  $G$  is  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ -measurable because  $g$  is  $\mathcal{K} \boxtimes \mathcal{F}$ -measurable. Next we claim that it is a Fubini extension of the usual triple product space  $((L \times K) \times \Omega, (\mathcal{L} \otimes \mathcal{K}) \otimes \mathcal{F}, (\eta \otimes \kappa) \otimes P)$ .

To show the Fubini property on the extended space, we adapt a proof analogous to the usual Fubini Theorem.<sup>9</sup> Let  $V \subseteq L^1(\eta \otimes (\kappa \boxtimes P))$  be the set of all  $\eta \otimes (\kappa \boxtimes P)$ -integrable function  $h$  satisfying the Fubini property. That is, (1)  $h_{(l, k)}$  is integrable on  $(\Omega, \mathcal{F}, P)$  for  $\eta \otimes \kappa$ -almost all  $(l, k) \in L \times K$  and  $h_\omega$  is integrable on  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$  for  $P$ -almost all  $\omega \in \Omega$ ; (2)  $\int_\Omega h_{(l, k)} dP$  and  $\int_{L \times K} h_\omega d\eta \otimes \kappa$  are integrable respectively on  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$  and  $(\Omega, \mathcal{F}, P)$ ; (3)  $\int_{L \times K \times \Omega} h d(\eta \otimes (\kappa \boxtimes P)) = \int_{L \times K} (\int_\Omega h_{(l, k)} dP) d\eta \otimes \kappa = \int_\Omega (\int_{L \times K} h_\omega d\eta \otimes \kappa) dP$ .

We shall first show that the set  $V$  contains all the indicator functions of the measurable sets in  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ . Let  $\mathcal{D}$  be the collection of all  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ -measurable sets  $D$  such that its indicator function  $1_D$  (which takes value 1 in  $D$  and 0 outside) is in  $V$ .

Consider  $D$  to be a measurable rectangle  $B \times W$  for  $B \in \mathcal{L}$  and  $W \in \mathcal{K} \boxtimes \mathcal{F}$ . The section  $D_\omega$  is  $B \times W_\omega$ . By the Fubini property associated with  $\kappa \boxtimes P$ ,  $B \times W_\omega$  is in  $\mathcal{L} \otimes \mathcal{K}$  for  $P$ -almost all  $\omega \in \Omega$ . The measure of  $D_\omega$  is  $\eta \otimes \kappa(D_\omega) = \eta(B)\kappa(W_\omega)$ , which is  $P$ -integrable with integral  $\eta(B) \int_\Omega \kappa(W_\omega) dP$ . Similarly, the section  $D_{(l, k)}$  is  $W_k$  if  $l \in B$ ,

---

and the space  $\{0, 1\}^\alpha$  with the cardinality  $\alpha$  between the cardinality  $c$  of the continuum and  $2^c$ . Thus the cardinality of  $\{0, 1\}^\alpha$  is at least  $2^c$  while the cardinality of our space  $K$  in Lemma 2 is  $c$ .

<sup>9</sup>See, for example, p. 308 of [22]. Similar adaption of the idea has been used in [13] to prove the one-way Fubini property.



and empty if  $l \notin B$ , which is in  $\mathcal{F}$  for  $\kappa$ -almost all  $k \in K$ . The measure of  $D_{(l,k)}$  is  $1_B(l)P(W_k)$ , which is  $\eta \otimes \kappa$ -integrable with integral  $\eta(B) \int_K P(W_k) d\kappa$ . By the Fubini property associated with  $\kappa \boxtimes P$  again,

$$\int_{\Omega} \eta(B) \kappa(W_{\omega}) dP = \eta(B) \int_{\Omega} \kappa(W_{\omega}) dP = \eta(B) \int_K P(W_k) d\kappa = \eta(B) (\kappa \boxtimes P)(W),$$

which means that

$$\int_{\Omega} \eta \otimes \kappa(D_{\omega}) dP = \int_{L \times K} P(D_{(l,k)}) d\eta \otimes \kappa = (\eta \otimes (\kappa \boxtimes P))(D).$$

Hence,  $B \times W \in \mathcal{D}$ .

Next, we show that the collection  $\mathcal{D}$  is also a Dynkin (or  $\lambda$ -) system on  $L \times K \times \Omega$ . Indeed, it is obvious that (i)  $L \times K \times \Omega \in \mathcal{D}$ ; (ii) if  $D, D' \in \mathcal{D}$  and  $D' \subseteq D$ , then  $D - D' \in \mathcal{D}$  because  $1_{D-D'} = 1_D - 1_{D'}$  and Fubini property is closed under linear combination; (iii) if  $D^n$  is an increasing sequence of sets in  $\mathcal{D}$ , then  $1_{D^n}$  is an increasing sequence of functions with limit  $1_{\cup_{n=1}^{\infty} D^n}$ , thus  $\cup_{n=1}^{\infty} D^n \in \mathcal{D}$  according to the Monotone Convergence Theorem (see the proof for a general sequence of functions below). Since the collection of measurable rectangles of the form  $B \times W$  for  $B \in \mathcal{L}$  and  $W \in \mathcal{K} \boxtimes \mathcal{F}$  is  $\pi$ -system (i.e., closed under finite intersections) and generates  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ , Dynkin's  $\pi$ - $\lambda$  Theorem (see p. 277 of [1] or p. 24 of [8]) implies that  $\mathcal{D} = \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ . Hence  $V$  contains all the indicator functions of the measurable sets in  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ .

As mentioned above, the set  $V$  is closed under linear combinations. In particular,  $V$  contains all the measurable simple functions and the difference between any two members. Note that each  $\eta \otimes (\kappa \boxtimes P)$ -integrable function is the difference between two non-negative integrable functions and each non-negative integrable function is the pointwise limit of an increasing sequence of non-negative simple functions. So we only need to show that for any increasing sequence of non-negative functions in  $V$  with an integrable pointwise limit, the limit function also belongs to  $V$ .

Now let  $h \in L^1(\eta \otimes (\kappa \boxtimes P))$ , and  $\{h^n\}_{n=1}^{\infty}$  be an increasing sequence of non-negative functions in  $V$  with pointwise limit  $h$  (to be denoted by  $h^n \uparrow h$ ). By the Monotone Convergence Theorem (see [22]),

$$\lim_{n \rightarrow \infty} \int_{L \times K \times \Omega} h^n d(\eta \otimes (\kappa \boxtimes P)) = \int_{L \times K \times \Omega} h d(\eta \otimes (\kappa \boxtimes P)).$$

Since  $h^n$  satisfies the Fubini property, we know that  $h_{\omega}^n$  is  $\eta \otimes \kappa$ -integrable for  $P$ -almost all  $\omega \in \Omega$ . For each  $\omega \in \Omega$ ,  $h_{\omega}^n \uparrow h_{\omega}$ . The Monotone Convergence Theorem implies that for  $P$ -almost all  $\omega \in \Omega$ ,  $\int_{L \times K} h_{\omega}^n d\eta \otimes \kappa \uparrow \int_{L \times K} h_{\omega} d\eta \otimes \kappa$ . By the Fubini property of  $h^n$  again,  $\int_{K \times L} h_{\omega}^n d\eta \otimes \kappa$  is  $P$ -integrable. Hence, the Monotone Convergence Theorem can be applied again to obtain that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left( \int_{L \times K} h_{\omega}^n d\eta \otimes \kappa \right) dP = \int_{\Omega} \left( \int_{L \times K} h_{\omega} d\eta \otimes \kappa \right) dP.$$

Since  $\int_{\Omega} \left( \int_{L \times K} h_{\omega}^n d\eta \otimes \kappa \right) dP = \int_{L \times K \times \Omega} h^n d(\eta \otimes (\kappa \boxtimes P))$ , we have

$$\int_{\Omega} \left( \int_{L \times K} h_{\omega} d\eta \otimes \kappa \right) dP = \int_{L \times K \times \Omega} h d(\eta \otimes (\kappa \boxtimes P)).$$

The other half of the Fubini property for  $h$  can be proved in a similar way. Hence  $h \in V$ .

Therefore, we show that  $V = L^1(\eta \otimes (\kappa \boxtimes P))$ , which means that the extended space  $(L \times K \times \Omega, \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F}), \eta \otimes (\kappa \boxtimes P))$  is a Fubini extension.

**Step 2.** Now consider a new process  $F$  from  $C \times \Omega$  to  $[0, 1]$  with  $F$  being the restriction  $G|_{C \times \Omega}$  of  $G$  to  $C \times \Omega$ , where  $G$  is the process in Step 1. The index probability space is restricted to  $(C, \mathcal{T}, \gamma)$  from the the product space  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$ , and the sample space  $(\Omega, \mathcal{F}, P)$  remains the same as in Step 1.

It is clear that for any  $(l, k) \in C$ ,  $F_{(l, k)}$  is a random variable on the sample space with uniform distribution  $\mu$  on  $[0, 1]$ . Moreover,  $F$  is an essentially pairwise independent process. In fact, for any  $(l_0, k_0) \in C \subseteq L \times K$ , the random variables  $F_{(l, k)} = G_{(l, k)}$  and  $F_{(l_0, k_0)} = G_{(l_0, k_0)}$  are independent for any  $(l, k) \in \{(l, k) \in L \times K : k \neq k_0\} \cap C$ , note that

$$\gamma[\{(l, k) \in L \times K : k \neq k_0\} \cap C] = \eta \otimes \kappa[\{(l, k) \in L \times K : k \neq k_0\}] = 1.$$

Recall that  $(L \times K \times \Omega, \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F}), \eta \otimes (\kappa \boxtimes P))$  is shown to be a Fubini extension in Step 1. We shall prove that  $C \times \Omega$  has  $\eta \otimes (\kappa \boxtimes P)$ -outer measure one. Let  $D$  be any measurable set in  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$  that contains  $C \times \Omega$ . Then, for each  $\omega \in \Omega$ ,  $C \subseteq D_\omega$ . By the Fubini property associated with  $\eta \otimes (\kappa \boxtimes P)$ , we have for  $P$ -almost all  $\omega \in \Omega$ ,  $D_\omega \in \mathcal{L} \otimes \mathcal{K}$ , and hence  $\eta \otimes \kappa(D_\omega) = 1$  since  $C$  has  $\eta \otimes \kappa$ -outer measure one. By the Fubini property associated with  $\eta \otimes (\kappa \boxtimes P)$  again,  $\eta \otimes (\kappa \boxtimes P)(D) = \int_\Omega \eta \otimes \kappa(D_\omega) dP = 1$ .

Based on the Fubini extension  $(L \times K \times \Omega, \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F}), \eta \otimes (\kappa \boxtimes P))$ , we can construct a measure structure on  $C \times \Omega$  as follows. Let  $\mathcal{E} = \{D \cap (C \times \Omega) : D \in \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})\}$  (which is a  $\sigma$ -algebra on  $C \times \Omega$ ), and  $\nu$  be the set function on  $\mathcal{E}$  defined by  $\nu(D \cap (C \times \Omega)) = \eta \otimes (\kappa \boxtimes P)(D)$  for any measurable set  $D$  in  $\mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ . Then,  $\nu$  is a well-defined probability measure on  $(C \times \Omega, \mathcal{E})$  since the  $\eta \otimes (\kappa \boxtimes P)$ -outer measure of  $C \times \Omega$  is one. It is obvious that the process  $F$  is  $\mathcal{E}$ -measurable.

Next, we show that the probability space  $(C \times \Omega, \mathcal{E}, \nu)$  extends the usual product probability space  $(C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes P)$ . Fix any  $Y \in \mathcal{T}$  and  $A \in \mathcal{F}$ . Then, there is a measurable set  $U \in \mathcal{L} \otimes \mathcal{K}$  such that  $Y = U \cap C$ . The rectangle  $Y \times A$  is  $(U \times A) \cap (C \times \Omega)$ ; and hence it belongs to  $\mathcal{E}$ . By the definitions of  $\gamma$  and  $\nu$ , we know that

$$\begin{aligned} (\gamma \otimes P)(Y \times A) &= \gamma(Y) \cdot P(A) = (\eta \otimes \kappa)(U) \cdot P(A) = (\eta \otimes \kappa \otimes P)(U \times A) \\ &= \eta \otimes (\kappa \boxtimes P)(U \times A) = \nu[(U \times A) \cap (C \times \Omega)] = \nu(Y \times A). \end{aligned}$$

Since  $\mathcal{E}$  contains all the rectangles  $Y \times A$  for  $Y \in \mathcal{T}$  and  $A \in \mathcal{F}$  (which generate  $\mathcal{T} \otimes \mathcal{F}$ ), it contains  $\mathcal{T} \otimes \mathcal{F}$ . Since the probability measures  $(\gamma \otimes P)$  and  $\nu$  agree on all the rectangles  $Y \times A$  for  $Y \in \mathcal{T}$  and  $A \in \mathcal{F}$  (which generate  $\mathcal{T} \otimes \mathcal{F}$  and form a  $\pi$ -system), the theorem on the uniqueness of measure (p. 404 of [8]) implies that  $(\gamma \otimes P)$  and  $\nu$  must agree on  $\mathcal{T} \otimes \mathcal{F}$ . Therefore,  $(C \times \Omega, \mathcal{E}, \nu)$  is an extension of  $(C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes P)$ .

In the following, we shall show that  $(C \times \Omega, \mathcal{E}, \nu)$  is a Fubini extension. Fix any measurable set  $E \in \mathcal{E}$ . Then,  $E = D \cap (C \times \Omega)$  for some  $D \in \mathcal{L} \otimes (\mathcal{K} \boxtimes \mathcal{F})$ .

For each  $\omega \in \Omega$ ,  $E_\omega = D_\omega \cap C$ . By the Fubini property associated with  $\eta \otimes (\kappa \boxtimes P)$ , we have for  $P$ -almost all  $\omega \in \Omega$ ,  $D_\omega$  is in  $\mathcal{L} \otimes \mathcal{K}$ , which means that  $E_\omega \in \mathcal{T}$  with  $\gamma(E_\omega) = \eta \otimes \kappa(D_\omega)$ . By the same Fubini property again,  $\eta \otimes \kappa(D_\omega)$  is  $P$ -integrable with integral  $\int_\Omega \eta \otimes \kappa(D_\omega) dP = \eta \otimes (\kappa \boxtimes P)(D)$ . Hence,  $\int_\Omega \gamma(E_\omega) dP = \eta \otimes (\kappa \boxtimes P)(D)$ , which implies that  $\int_\Omega \gamma(E_\omega) dP = \nu(E)$  by the definition of  $\nu$ .

Next we shall prove the other part of the Fubini property associated with  $\nu$  for the measurable set  $E \in \mathcal{E}$ . Recall that  $\nu(E) = \eta \otimes (\kappa \boxtimes P)(D)$ . By the Fubini property associated with  $\eta \otimes (\kappa \boxtimes P)$ , the function  $P(D_{(l,k)})$  on  $L \times K$  is integrable over  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$  with integral  $\eta \otimes (\kappa \boxtimes P)(D) = \int_{L \times K} P(D_{(l,k)}) d\eta \otimes \kappa$ . Since  $(L \times K, \mathcal{U}, \gamma)$  is an extension of  $(L \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$ ,  $P(D_{(l,k)})$  is integrable over  $(L \times K, \mathcal{U}, \gamma)$  with  $\int_{L \times K} P(D_{(l,k)}) d\gamma = \int_{L \times K} P(D_{(l,k)}) d\eta \otimes \kappa$ . Since  $C \in \mathcal{U}$  with measure  $\gamma(C) = 1$  and  $\mathcal{T}$  is the restriction of  $\mathcal{U}$  to  $C$ , the restriction of  $P(D_{(l,k)})$  to  $C$  is integrable over  $(C, \mathcal{T}, \gamma)$  with  $\int_C P(D_{(l,k)}) d\gamma = \int_{L \times K} P(D_{(l,k)}) d\gamma$ . Since  $E_{(l,k)} = D_{(l,k)}$  for any  $(l, k) \in C$ , we know that  $P(E_{(l,k)})$  is integrable over  $(C, \mathcal{T}, \gamma)$  with  $\int_C P(E_{(l,k)}) d\gamma = \int_C P(D_{(l,k)}) d\gamma$ . By combining all these equalities together, we obtain that  $\nu(E) = \int_C P(E_{(l,k)}) d\gamma$ .

Therefore the indicator function  $1_E$  satisfies the Fubini property for any measurable set  $E \in \mathcal{E}$ . The rest of the proof of the Fubini property is the same as in Step 1. Thus the probability space  $(C \times \Omega, \mathcal{E}, \nu)$  is a Fubini extension of the usual product probability space  $(C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes P)$ .

**Step 3.** Now let  $f : I \times \Omega \rightarrow [0, 1]$  be another process defined by  $f(i, \omega) = F(\psi^{-1}(i), \omega)$  for any  $(i, \omega) \in I \times \Omega$ , where  $F$  is the process in Step 2 and  $\psi$  the isomorphism between the probability spaces  $(C, \mathcal{T}, \gamma)$  and  $(I, \mathcal{I}, \lambda)$ . It is clear that the process  $f$  is essentially pairwise independent and the random variable  $f_i$  has uniform distribution  $\mu$  on  $[0, 1]$  for any  $i \in I$ .

Given the Fubini extension  $(C \times \Omega, \mathcal{E}, \nu)$  of the usual product probability space  $(C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes P)$  in Step 2, we can use the bijection  $(\psi, \text{Id}_\Omega)$  from  $C \times \Omega$  to  $I \times \Omega$  to construct a  $\sigma$ -algebra  $\mathcal{W} = \{H \subseteq I \times \Omega : (\psi, \text{Id}_\Omega)^{-1}(H) \in \mathcal{E}\}$  on  $I \times \Omega$ , where  $\text{Id}_\Omega$  is the identity map on  $\Omega$ . Define a probability measure  $\rho$  on  $\mathcal{W}$  by letting  $\rho(H) = \nu[(\psi, \text{Id}_\Omega)^{-1}(H)]$  for any  $H \in \mathcal{W}$ . Therefore,  $(\psi, \text{Id}_\Omega)$  is also an isomorphism between the two probability spaces  $(C \times \Omega, \mathcal{E}, \nu)$  and  $(I \times \Omega, \mathcal{W}, \rho)$ . The process  $f$  is obviously  $\mathcal{W}$ -measurable.

For  $S \in \mathcal{I}$  and  $Y \in \mathcal{F}$ , the definition of  $\mathcal{I}$  implies that  $\psi^{-1}(S) \in \mathcal{T}$ , and hence  $(\psi, \text{Id}_\Omega)^{-1}(S \times Y) \in \mathcal{T} \otimes \mathcal{F} \subseteq \mathcal{E}$ . Therefore, the definition of  $\mathcal{W}$  implies that  $S \times Y \in \mathcal{W}$ . By the definition of  $\rho$ ,

$$\begin{aligned} \rho(S \times Y) &= \nu[(\psi, \text{Id}_\Omega)^{-1}(S \times Y)] = \nu[\psi^{-1}(S) \times Y] = \gamma \otimes P[\psi^{-1}(S) \times Y] \\ &= \gamma[\psi^{-1}(S)] \cdot P(Y) = \lambda(S)P(Y) = \lambda \otimes P(S \times Y). \end{aligned}$$

The probability measures  $\rho$  and  $\lambda \otimes P$  agree on all the rectangles  $S \times Y$  for  $S \in \mathcal{I}$  and  $Y \in \mathcal{F}$ , which generate  $\mathcal{I} \otimes \mathcal{F}$  and form a  $\pi$ -system. As in Step 2, the theorem on the uniqueness of measure (p. 404 of [8]) implies that  $(\lambda \otimes P)$  and  $\rho$  must agree on  $\mathcal{I} \otimes \mathcal{F}$ . Therefore,  $(I \times \Omega, \mathcal{W}, \rho)$  is an extension of  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ .

Next, we prove the Fubini property associated with  $\rho$ . As in Step 2, we only prove this property for any measurable set  $H \in \mathcal{W}$ . Let  $E = (\psi, \text{Id}_\Omega)^{-1}(H)$ ; then  $E \in \mathcal{E}$  and  $\nu(E) = \nu[(\psi, \text{Id}_\Omega)^{-1}(H)] = \rho(H)$ .

It is obvious that for any  $\omega \in \Omega$ ,  $\psi^{-1}(H_\omega) = E_\omega$ , and  $H_\omega = \psi(E_\omega)$ . By the definition of  $\lambda$ ,  $\lambda(H_\omega) = \gamma[\psi^{-1}(H_\omega)] = \gamma(E_\omega)$ . By the Fubini property associated with  $\nu$  for  $E$ , for  $P$ -almost all  $\omega \in \Omega$ ,  $E_\omega \in \mathcal{T}$ , and thus it follows from the definition of  $\mathcal{I}$  that  $H_\omega = \psi(E_\omega) \in \mathcal{I}$ . By the Fubini property associated with  $\nu$  for  $E$  again, the  $P$ -integrable function  $\gamma(E_\omega)$  has integral  $\nu(E) = \int_\Omega \gamma(E_\omega) dP$ . Since  $\rho(H) = \nu(E)$ , we have  $\rho(H) = \int_\Omega \lambda(H_\omega) dP$ .

For the other part of the Fubini property associated with  $\rho$  for  $H$ , note that  $H_i = E_{\psi^{-1}(i)}$  for each  $i \in I$ . By the Fubini property of  $\nu$  for  $E$ , there is a set  $T \in \mathcal{T}$  with  $\gamma(T) = 1$  such that for any  $(l, k) \in T$ ,  $E_{(l, k)} \in \mathcal{F}$ . Hence,  $\psi(T) \in \mathcal{I}$  with  $\lambda(\psi(T)) = 1$ , and for each  $i \in \psi(T)$ ,  $H_i = E_{\psi^{-1}(i)} \in \mathcal{F}$ . By the Fubini property of  $\nu$  for  $E$  and the formula for changing variables,

$$\begin{aligned} \nu(E) &= \int_C P(E_{(l, k)}) d\gamma = \int_T P(E_{(l, k)}) d\gamma = \int_{\psi(T)} P(E_{\psi^{-1}(i)}) d\lambda \\ &= \int_{\psi(T)} P(H_i) d\lambda = \int_I P(H_i) d\lambda. \end{aligned}$$

Since  $\rho(H) = \nu(E)$ , we have  $\rho(H) = \int_I P(H_i) d\lambda$ .

Therefore,  $(I \times \Omega, \mathcal{W}, \rho)$  is a Fubini extension. As in Definition 3, we denote  $(I \times \Omega, \mathcal{W}, \rho)$  by  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ .  $\square$

In the following lemma, we restate Proposition 5.3 of [23].

**Lemma 4** *Let  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  be a Fubini extension, and  $X$  a complete separable metric space. Assume that there exists an essentially pairwise independent process  $f$  from  $I \times \Omega$  to  $[0, 1]$  such that  $f$  is  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable, and for each  $i \in I$ , the distribution of the random variable  $f_i$  is the uniform distribution  $\mu$  on  $[0, 1]$ .<sup>10</sup> Then, for any measurable mapping  $\varphi$  from  $(I, \mathcal{I}, \lambda)$  to the space  $\mathcal{M}(X)$  of Borel probability measures on  $X$ , there is a  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process  $g$  from  $I \times \Omega$  to  $X$  such that the random variables  $g_i$  are essentially pairwise independent, and the distribution  $Pg_i^{-1}$  is the given distribution  $\varphi(i)$  for  $\lambda$ -almost all  $i \in I$ .*

**Proof of the Theorem 1:** It is now obvious that Theorem 1 follows from Proposition 1 and Lemma 4.  $\square$

## References

- [1] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer, Berlin, 1994.
- [2] R.M. Anderson, Non-standard analysis with applications to economics, in *Handbook of Mathematical Economics*, Volume IV (W. Hildenbrand and H. Sonnenschein Eds.), North-Holland, New York, 1991.
- [3] R.J. Aumann, Markets with a continuum of traders, *Econometrica* **32** (1964), 39-50.
- [4] J.L. Doob, Stochastic processes depending on a continuous parameter, *Transactions of the American Mathematical Society* **42** (1937), 107-140.
- [5] J.L. Doob, *Stochastic Processes*, Wiley, New York, 1953.
- [6] D. Duffie, N. Garleanu and L.H. Pedersen, Over-the-counter markets, *Econometrica* **73** (2005), 1815-1847.

---

<sup>10</sup>Such a Fubini extension is called a *rich product probability space* in [23].

- [7] D. Duffie, N. Garleanu and L.H. Pedersen, Valuation in over-the-counter markets, *Review of Financial Studies* **20** (2007), 1865-1900.
- [8] R. Durrett, *Probability: Theory and Examples*, 3<sup>rd</sup> edition, Brooks/Cole, Belmont, 2005.
- [9] M. Feldman and C. Gilles, An expository note on individual risk without aggregate uncertainty, *Journal of Economic Theory* **35** (1985), 26-32.
- [10] D.H. Fremlin, *Measure Theory, Volume 4: Topological Measure Spaces*, Torres Fremlin, Colchester, 2003.
- [11] D.H. Fremlin, *Measure Theory, Volume 5: Set-theoretic Measure Theory (version 8.5.03/29.6.06)*, 2005. Available at <http://www.essex.ac.uk/maths/staff/fremlin/mt.htm>
- [12] E.J. Green, Individual-level randomness in a nonatomic population, University of Minnesota, mimeo., 1994.
- [13] P.J. Hammond and Y.N. Sun, Joint measurability and the one-way fubini property for a continuum of independent random variables, *Proceedings of American Mathematical Society* **134** (2006), 737-747.
- [14] K.L. Judd, The law of large numbers with a continuum of i.i.d. random variables, *Journal of Economic Theory* **35** (1985), 19-25.
- [15] H.J. Keisler and Y.N. Sun, Loeb measures and Borel algebras, in *Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum* (Ulrich Berger, Horst Osswald and Peter Schuster eds.), Kluwer Academic Publishers, Dordrecht, 2001, pp. 111-117.
- [16] M.A. Khan, Perfect competition, to appear in *The New Palgrave*, mimeo., Johns Hopkins University, 2007.
- [17] R. Lagos and G. Rocheteau, Liquidity in asset markets with search frictions, NYU Working Paper, 2007.
- [18] P.A. Loeb and M. Wolff, eds. *Nonstandard Analysis for the Working Mathematician*, Kluwer Academic Publishers, Dordrecht, 2000.
- [19] R. McLean and A. Postlewaite, Informational size and incentive compatibility, *Econometrica* **70** (2002), 2421-2453.
- [20] R. McLean and A. Postlewaite, Core convergence with asymmetric information, *Games and Economic Behavior* **50** (2005), 58-78.
- [21] K. Podczeck, On the convexity and compactness of the integral of a Banach space valued correspondence, *Journal of Mathematical Economics*, published online.
- [22] H.L. Royden, *Real Analysis*, Macmillan, New York, 1968.
- [23] Y.N. Sun, The exact law of large numbers via Fubini extension and characterization of insurable risks, *Journal of Economic Theory* **126** (2006), 31-69.
- [24] Y.N. Sun and N.C. Yannelis, Core, equilibria and incentives in large asymmetric information economies, *Games and Economic Behavior* **61** (2007), 131-155.
- [25] P.-O. Weill, Leaning against the wind, *Review of Economic Studies* **74** (2007), 1329-1354.